

Resonance in a series RLC circuit

Theoretical background

1. Alternating current and voltage

An alternating current (AC) is an electrical current where the magnitude and direction of the current varies cyclically, so as the time average is zero (as opposed to direct current (DC), where the direction of the current stays constant). The usual waveform of an AC power circuit is a sine wave. (However, in certain applications different waveforms are used, such as triangular or square waves.)

The generators in the power plants produce sinusoidal voltages. The current flowing through the electric appliances using AC power supply is also sinusoidal. The AC voltages and currents are characterized by their

frequency f , or angular frequency $\omega = 2\pi f$,

amplitude U_0 and I_0 , respectively, and

phase constant φ_U and φ_I , respectively:

$$U(t) = U_0 \cos(\omega t + \varphi_U) \quad \text{and} \quad I(t) = I_0 \cos(\omega t + \varphi_I).$$

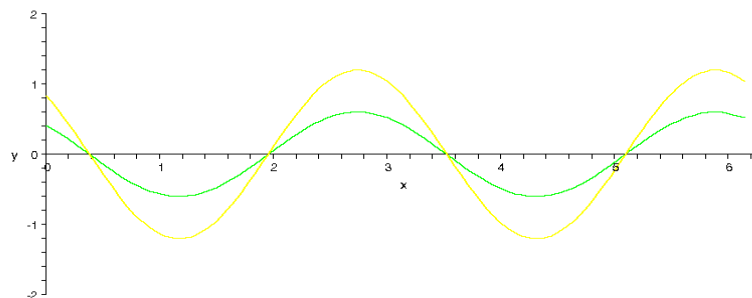
The voltage and the current are not necessarily in-phase, there can be some phase difference $\Delta\varphi = \varphi_U - \varphi_I$ between them. Choosing the phase constant of the current zero, the time dependence of the current and voltage in an AC circuit can be written as

$$I(t) = I_0 \cos(\omega t) \quad \text{and} \quad U(t) = U_0 \cos(\omega t + \Delta\varphi).$$

2. Resistor in an AC circuit

Ohm's Law is valid for the time-dependent voltage and current so

$$U_R(t) = R \cdot I(t)$$



green: $I(t)$, yellow: $U_R(t)$

Across a resistor the voltage and the current are in-phase ($\Delta\varphi = 0$),
and for the amplitudes holds $U_{R,0} / I_{R,0} = R$.

3. Capacitor in an AC circuit

The voltage across the capacitor is proportional to the charge Q accumulated on its plates:

$$U_C(t) = \frac{1}{C} \cdot Q(t) \quad \text{where } C \text{ is the capacitance in Farad [F].}$$

As the current is the derivative of the charge $I(t) = \frac{dQ(t)}{dt}$

so the integral of the current is proportional to the voltage

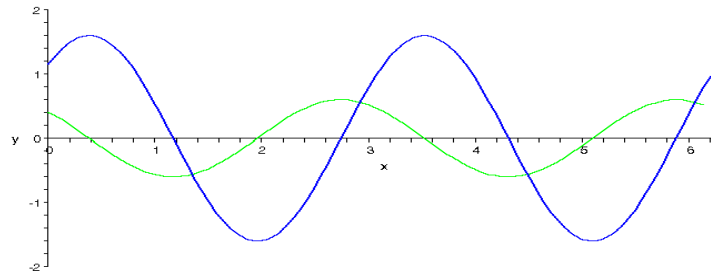
$$U_C(t) = \frac{1}{C} \cdot \int I(t) dt.$$

In case of a sinusoidal current $I(t) = I_0 \cos(\omega t)$

$$U_C(t) = \frac{1}{C} \cdot \int I_{C0} \cos(\omega t) dt = \frac{1}{\omega C} \cdot I_{C0} \sin(\omega t) = \frac{1}{\omega C} \cdot I_{C0} \cos\left(\omega t - \frac{\pi}{2}\right).$$

There will be a phase shift between the voltage and the current: $\Delta\varphi = -\pi/2$,

and $U_{C0} = \frac{1}{\omega C} \cdot I_{C0}$, i.e. $\frac{U_{C0}}{I_{C0}} = \frac{1}{\omega C}$.



green: $I(t)$, blue: $U_C(t)$

Across a capacitor the phase difference is $\Delta\varphi = -\pi/2$ (the voltage is delayed),

and for the amplitudes holds $U_{C,0} / I_{C,0} = \frac{1}{\omega C}$.

4. Inductor in an AC circuit

In case of a time-dependent current a strong magnetic field is generated inside the coil due to Ampere's Law, and due to the time-varying magnetic field inside the coil a voltage is induced, according to Faraday's Law of electromagnetic induction (which by Lenz's Law opposes the change in current that created it). So the voltage across the coil is proportional to the derivative of the current flowing through it

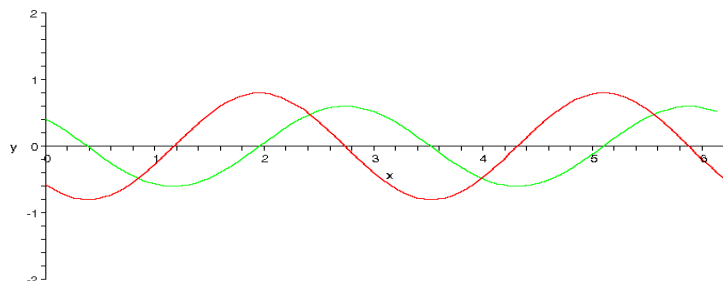
$$U_L(t) = L \cdot \frac{dI(t)}{dt} \quad \text{where } L \text{ is the inductance in Henry [H].}$$

In case of a sinusoidal current $I(t) = I_0 \cos(\omega t)$

$$U_L(t) = L \cdot \frac{I_{L0} \cos(\omega t)}{dt} = -L\omega \cdot I_{L0} \sin(\omega t) = L\omega \cdot I_{L0} \cos\left(\omega t + \frac{\pi}{2}\right).$$

There will be a phase shift between the voltage and the current: $\Delta\varphi = +\pi/2$,

and $U_{L0} = L\omega \cdot I_{L0}$, i.e. $\frac{U_{L0}}{I_{L0}} = L\omega$.



green: $I(t)$, red: $U_L(t)$

Across a coil the phase difference is $\Delta\varphi = +\pi/2$ (the current is delayed),

and for the amplitudes holds $U_{L,0} / I_{L,0} = \omega L$.

5. Using complex quantities for AC circuits

Instead of using the sine and cosine functions, it is simpler to treat AC circuits if we represent AC currents and voltages by complex quantities:

$$\tilde{I} = I_0 e^{j\omega t} \quad , \quad \tilde{U} = U_0 e^{j\omega t + \Delta\varphi} \quad .$$

The real time-dependent voltage or current is obtained as the real part of the complex form. The ratio of the real voltage and current would change with time in an AC circuit, but it is constant in the complex representation. The ratio of the complex voltage to the complex current is called **complex impedance** and denoted by \tilde{Z} :

$$\tilde{Z} = Z e^{j\varphi} = \frac{\tilde{U}}{\tilde{I}} = \frac{U_0 e^{j\omega t + \Delta\varphi}}{I_0 e^{j\omega t}} = \frac{U_0}{I_0} e^{j\Delta\varphi} \quad .$$

The magnitude of the complex impedance $Z = |\tilde{Z}| = \frac{U_0}{I_0}$ is equal to the ratio of the amplitude of the voltage to the amplitude of the current, and

the phase of the complex impedance φ is equal to the phase difference between voltage and current, $\varphi = \Delta\varphi = \varphi_U - \varphi_I$.

6. Complex impedance of resistors, capacitors and inductors

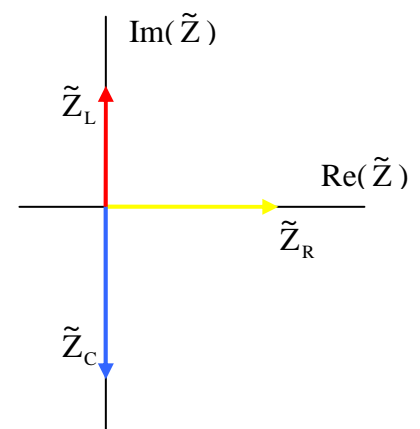
The impedance of a resistor is equal to the resistance: $\tilde{Z}_R = R$. There is no phase difference between the voltage and the current.

The impedance of a capacitor is $\tilde{Z}_C = \frac{1}{j\omega C} = -j \frac{1}{\omega C}$.

The voltage of a capacitor lags behind the current by $\pi/2$.

The impedance of an inductor is $\tilde{Z}_L = j\omega L$.

The voltage of an inductor leads the current by $\pi/2$.



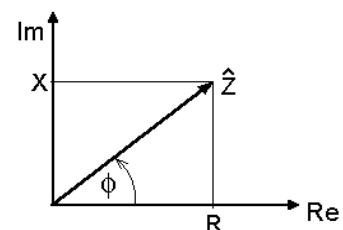
7. Resultant impedance

Making a two-pole by connecting resistors, capacitors and inductors in any way, it can be replaced by a resultant impedance. Though the impedance of the constructing elements is either real or imaginary, the resultant impedance in a general case has both a real and an imaginary part: $\tilde{Z} = R + jX$.

We call the real part “resistance”, R and the imaginary part “reactance”, X .

The magnitude Z and phase φ of the impedance are related to R and X as follows:

$$Z = \sqrt{R^2 + X^2} \quad , \quad \text{and} \quad \tan(\varphi) = \frac{X}{R} \quad .$$



The resultant impedance is calculated similarly to the resultant resistance:

in a series connection $\tilde{Z}_{res,series} = \tilde{Z}_1 + \tilde{Z}_2 + \dots$,

in a parallel connection $\frac{1}{\tilde{Z}_{res,parallel}} = \frac{1}{\tilde{Z}_1} + \frac{1}{\tilde{Z}_2} + \dots$.

According to the resultant complex impedance the real AC voltage and current are related so that

- the ratio of the amplitudes (or rms values) is given by the magnitude of the impedance, and
- the phase difference between the voltage and the current is equal to the phase of the impedance.

The AC power dissipated on an impedance is

$$P = U_{rms} I_{rms} \cos \varphi = Z I_{rms}^2 \cos \varphi.$$

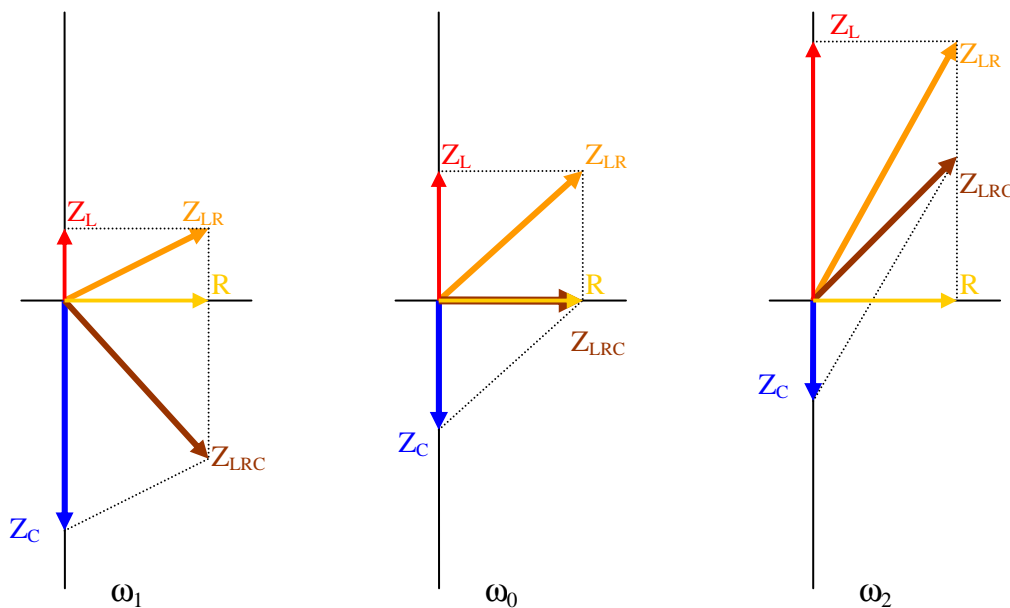
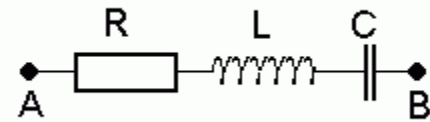
On an inductor or capacitor, the phase is $\varphi = \pm \pi/2$, and the power is zero.

Only the resistors dissipate power.

8. Series RLC circuit

The complex impedance of a series RLC circuit at angular frequency ω is

$$\tilde{Z}_{RLC} = \tilde{Z}_R + \tilde{Z}_L + \tilde{Z}_C = R + j\omega L - j\frac{1}{\omega C} = R + j\left(\omega L - \frac{1}{\omega C}\right).$$



Complex impedance of a **capacitor**, **real coil** and **their resultant** as the angular frequency increases, $\omega_1 < \omega_0 < \omega_2$.

The magnitude of the impedance

$$Z_{RLC} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

takes its minimal value at $\omega = \omega_0$ when $\omega_0 L = \frac{1}{\omega_0 C}$.

The frequency where this holds is called the **resonant frequency** f_0 of the RLC circuit.

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}} \quad \text{Thomson's formula}$$

At the resonant frequency $Z_{RLC} = R$.

As we know, the magnitude of the complex impedance is equal to the ratio of the amplitude of the voltage to that of the current, or – instead of the amplitude – the rms values can be also used:

$$Z = |\tilde{Z}| = \frac{U_0}{I_0} = \frac{U_{rms}}{I_{rms}}.$$

In a series RLC circuit

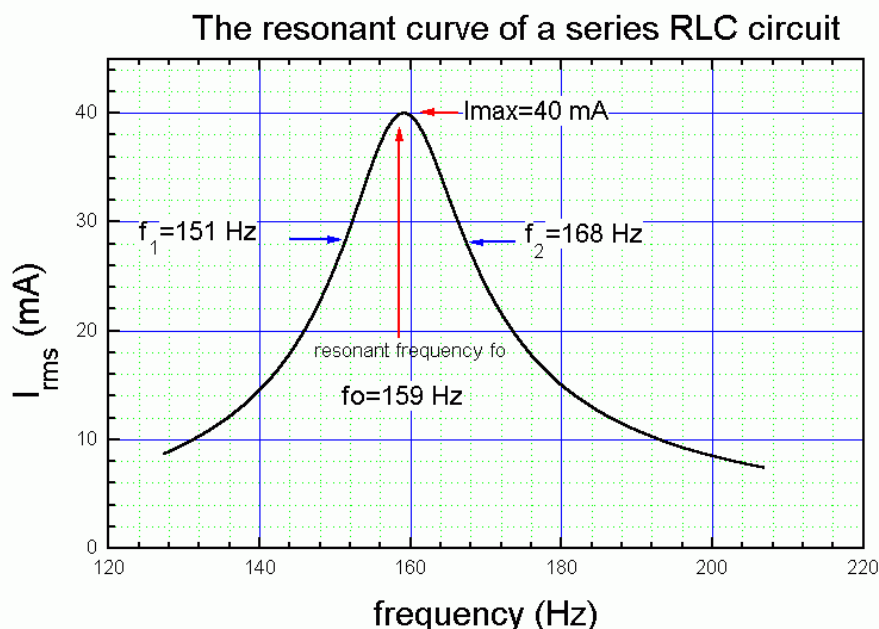
$$\frac{U_{rms}}{I_{rms}} = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}.$$

Driving the RLC circuit with a constant-rms value AC voltage and increasing the angular frequency, the current will increase first, reach a maximum and then decrease, according to the function

$$I_{rms} = \frac{U_{rms}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}.$$

The maximum current is $I_{rms} = \frac{U_{rms}}{R}$ at the resonant frequency ω_0 .

Such a resonant curve is shown in the figure below.



Here $U_{rms} = 4.0 \text{ V}$.

What data of the circuit can be obtained from such a resonant curve? We can read

- the maximum current I_{max} ; here $I_{max} = 40 \text{ mA}$;
- the resonant frequency f_0 ; here $f_0 = 159 \text{ Hz}$;
- f_1 and f_2 , the frequencies where the current is $\frac{I_{max}}{\sqrt{2}}$; here $f_1 = 151 \text{ Hz}$, $f_2 = 168 \text{ Hz}$.

From these data we can get

- the resistance: $R = \frac{U_{rms}}{I_{rms}}$; here $R = 4.0/40 \cdot 10^{-3} = 100 \, \Omega$;
- the quality factor: $Q = \frac{f_0}{f_2 - f_1}$; here $Q = 159/(168-151) = 9.353$.

The **Q quality factor** characterizes the sharpness of the resonance. It can be also calculated as

$$Q = \frac{1}{R} \sqrt{\frac{L}{C}} \quad \text{or} \quad Q = \frac{\omega_0 L}{R} \quad \text{or} \quad Q = \frac{1}{\omega_0 C R},$$

as it is equal to the voltage across a reactive element at the resonant frequency divided by the voltage across the whole circuit at the resonant frequency.

One possibility to calculate the inductance L and the capacitance C is to use these formulas and substitute the values of R , Q and $\omega_0 = 2\pi f_0$:

$$\text{here } L = \frac{QR}{2\pi f_0} = 0.9359 \, \text{H}, \text{ and } C = \frac{1}{2\pi f_0 QR} = 1.070 \, \mu\text{F}.$$

Note that one of them could be also calculated from the resonant frequency as $LC = \frac{1}{(2\pi f_0)^2}$.

L and C can be also calculated by measuring the current, and the voltage separately across the inductor and the coil, and use the formulas of the impedances.

Measurement

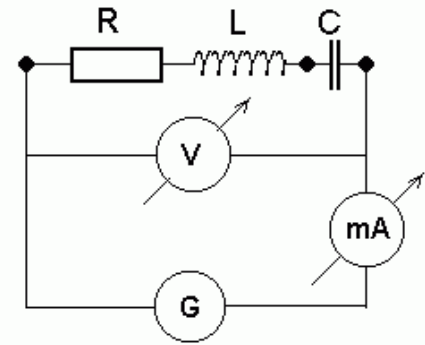
Tools:

AC generator, capacitor, inductor, ammeter, voltmeter, connecting wires.

The inductor –a solenoid equipped with an iron core, used to ignite light tubes– has its own resistance in addition to the inductance. It can be considered as an ideal inductor and a resistor connected in series: $\tilde{Z}_{RL} = R + j\omega L$.

Set up the series RLC circuit from the AC generator, capacitor, inductor, and ammeter.

Connect the voltmeter in parallel with the capacitor and the inductor.



Set the frequency and measure the current, the voltage across the RLC circuit, the voltage across the capacitor, and the voltage across the inductor, as indicated in the data sheet.

Evaluation

Calculate $U_{LR} + U_C$ for $f = 800$ Hz, 1000 Hz, 1200 Hz and compare it with the measured U_{LRC} . Explain the results.

First correct the measured current for $U_{LRC} = 6,0$ V (in Excel).

Plot the corrected current I vs. the frequency f on an A4-size mm paper.

Find the resonant frequency f_0 (the frequency for the highest current I_{\max}), and the frequencies needed to calculate the quality factor.

Calculate the ohmic resistance of the inductor R as $R = \frac{U_{rms}}{I_{rms}}$ at the resonant frequency.

Plot the impedance of the capacitor $Z_C = U_C / I$ vs. the frequency f (in Excel).

Calculate the capacitance C using SciDAVis:

$$\frac{U_C}{I} = \frac{1}{2\pi C} \cdot \frac{1}{f}$$

Plot the impedance of the inductor $Z_{LR} = U_{LR} / I$ vs. the frequency f (in Excel).

Calculate the inductivity L using SciDAVis (the value of R is already known):

$$\frac{U_{LR}}{I} = \sqrt{R^2 + (2\pi L)^2 \cdot f^2}$$

Calculate the resonant frequency using the Thomson formula, and compare it with the value read from the I vs. f graph.

Calculate the value of the quality factor Q using the formulas $Q = \frac{1}{R} \sqrt{\frac{L}{C}}$ and $Q = \frac{f_0}{f_2 - f_1}$.