1. Errors of measurement data. Confidence interval

A quantity – unless it is countable – cannot be given with absolute accuracy. The value given as the measure of the quantity will be always different from the real value. The deviation of the given value from the real one is the ***error***.
The most inherent error comes from the representation of numbers: we can give a number only with finite number of digits, and read the measurement results from a meter also with a finite number of digits. This ***reading error*** cannot be avoided. We take 1 unit of the last significant digit of a measurement or of a given quantity as the error if the error is not otherwise specified.
Measuring a quantity, the measured value can deviate from the real one more than the reading error. The magnitude of the deviation is called the error of the measured value.
The ***absolute error*** Δx of a quantity x is $\Delta x = |x_{measured} - x_{real}|$.
The ***relative error*** δx of the same quantity is $\delta x = \Delta x / x_{real}$. The relative error is usually given in percents.
The error can be caused e.g. by a faulty meter. To avoid this ***systematic error*** caused by the measuring apparatus, all meters have to be checked regularly and be calibrated against some known standard.
Even with a well calibrated meter and a careful measurement procedure we might get different results if we repeat the measurement. When the fluctuations of measurement results are caused by random effects we speak about ***accidental errors***. Such random effects can arise from the measured quantity which might not have the same value at every instant, or from the measuring apparatus, or from the observer.
The accidental errors obey the laws of probability theory. The possible result of the measurement is a random variable with a probability distribution. Usually it is supposed a ***Gaussian distribution***, with probability density of

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$

μ is the ***expectation value (mean, population mean)*** of the distribution, and
σ is the ***standard deviation***, the “half-width” of the probability density curve.
The probability is the integral of the above function; $x \in [\mu \pm \sigma]$ with P=68.3%; $x \in [\mu \pm 2\sigma]$ with P=95.4%; $x \in [\mu \pm 3\sigma]$ with P=99.7%.

We assume that the measurement results fluctuate around the “real value”, μ, and we estimate it by the ***arithmetic mean (sample mean)*** $\bar{x}$ of all the data obtained when we repeated the measurement N times:

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i}{N}.$$

The fluctuation of the measurement results, characterized by the standard deviation σ is estimated by the ***experimental standard deviation (or standard error)*** s_x:

$$s_x = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \bar{x})^2}{N-1}}.$$

This standard deviation characterizes both the measurement procedure and tool, and also the ensemble from where a sample was taken.
But it is also important to know, how much our measurement result $\bar{x}$ might deviate from the real expectation value μ. The probable deviation of $\bar{x}$ from μ is called the ***standard deviation of the mean***, $\boldsymbol{\sigma_{\bar{x}}}$:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}} \,,$$

and it is estimated from the data of the repeated measurement as

$$s_{\bar{x}} = \frac{s_x}{\sqrt{N}} = \sqrt{\frac{\sum_{i=1}^{N}(x_i-\bar{x})^2}{N(N-1)}} \,.$$

$\boldsymbol{s_{\bar{x}}}$, the ***experimental standard deviation of the mean***, decreases with the number of measurements. We can make the measurement more accurate if we repeat it and calculate the mean of the results.

We know $\bar{x}$ and $s_{\bar{x}}$ from the measurement results, but it is more practical to know with some certainty how far the mean $\bar{x}$ can be from the real value μ. The interval around the given value $\bar{x}$ that contains the real value μ with *P* probability is called the ***confidence interval*** or ***error interval***, and ***P*** is called the ***confidence level***. We get the radius of this interval, Δx, by using the so-called ***Student's t parameter***, ***t***, given for a specified level of confidence *P* and number of repeated measurements *N*.
The radius of the error interval is

$$\Delta x = t \cdot s_{\bar{x}} \,.$$

The parameter *t* can be found in tables for Student's distribution (see below).

Confidence interval (error interval)

We give the results of a repeated measurement in the form

$$x = \ (\bar{x} \pm \Delta x) \ \text{[unit]} \,,$$

where
$\bar{x}$ is the arithmetic mean of the results of measurement repeated *N* times, and
Δx is the error at a specified level of confidence *P* (usually 95%);

$\Delta x = t \cdot s_{\bar{x}}$, with $s_{\bar{x}} = \sqrt{\frac{\sum_{i=1}^{N}(x_i-\bar{x})^2}{N(N-1)}}$ and *t* the Student's parameter.

Tolerance means the allowed deviation from the nominal value of a given quantity. The tolerance is usually given for *P* = 95% confidence level. E.g. if a resistor has the nominal value of 200 Ω and a tolerance of 10%, than the actual resistance should be with *P* = 95% probability between 180 Ω and 220 Ω; i.e. in a pile of resistors 95% should be in this interval.

Values of *t* (Student's parameter)

at *N* number of repeated measurement data and at *P* confidence level

N \ P	**0.80**	**0.90**	**0.95**	**0.975**	**0.99**	**0.995**
2	3.078	6.314	12.706	25.452	63.657	127.32
3	1.886	2.920	4.303	6.205	9.925	14.089
4	1.638	2.353	3.182	4.176	5.841	7.453
5	1.553	2.132	2.776	3.495	4.604	5.598
6	1.476	2.015	2.571	3.163	4.032	4.773
7	1.440	1.943	2.447	2.969	3.707	4.317
8	1.415	1.895	2.365	2.841	3.499	4.029
9	1.397	1.860	2.306	2.752	3.355	3.832
10	1.383	1.833	2.262	2.685	3.250	3.690
20	1.328	1.729	2.093	2.433	2.861	3.174
∞	1.282	1.645	1.960	2.241	2.576	2.807

Example

We have a pile of resistors with the same nominal resistance which is unknown for us. We take 6 resistors and measure their resistance:

98 Ω   100 Ω   101 Ω   99 Ω   101 Ω   101 Ω

Calculate the nominal resistance, and the error interval at *P* = 99% confidence level!

Calculation:

The nominal resistance is calculated as the arithmetic mean: $\bar{R}$ = 100 Ω.

Then the standard deviation of the mean is calculated:

$$s_{\bar{R}} = \sqrt{\frac{(98-100)^2+(100-100)^2+(101-100)^2+(99-100)^2+(101-100)^2+(101-100)^2}{6 \cdot 5}} = \sqrt{\frac{8}{30}} \approx 0.5164\ \Omega,$$

and the value of the Student's parameter is read from the table:

for *N* = 6 and *P* = 0.99 the Student's parameter is t = 4.032.

The error interval is calculated as

$\Delta R = t \cdot s_{\bar{R}}$ = 4.032 · 0.5164 ≈ 2.082 Ω.

So the value of the resistances with P = 99% probability is in the interval

R = (100.0 ± 2.1) Ω

Note:

The error interval is given with 2 significant figures, and

the mean is given with the same accuracy as the error.

2. Propagation of errors

Sometimes a quantity cannot be measured directly, but it depends on other quantities which are easier to measure. For example, the average speed of a car is determined from the distance *D* it travelled and the time *t* needed to reach so far. Both the distance and the time are easy to measure and the speed is calculated as $v = D / t$.
We can measure the distance and time with errors ΔD and Δt, respectively. What will be the error of the speed?
We know that a slight change in one of the variables *x,y,…* of a function *f(x,y,…)* causes a change in the function approximately equal to the partial derivative with respect to that variable, multiplied by the change of the variable: $\Delta f_x \approx \frac{\partial f}{\partial x} \cdot \Delta x$.
That is true for the errors, too. The error of the function *f* caused by the error of one of its variables *x* is

$$|\Delta f_x| = \left| \frac{\partial f}{\partial x} \cdot \Delta x \right|.$$

If there are more variables, the errors caused by all of them can be *summarized*; in this case we get *the maximum error*. But this maximum error is an overestimation, and it is used only if we want to give our results with maximum confidence.
Usually the errors are random, some will increase the quantity *f*, and some might decrease it.

Error propagation formula

We get the probable error Δf of a function *f* from the errors of its variables *x, y, …* as

$$\Delta f = \sqrt{\left(\frac{\partial f}{\partial x} \cdot \Delta x\right)^2 + \left(\frac{\partial f}{\partial y} \cdot \Delta y\right)^2 + \cdots}\ .$$

Δx, Δy, … are the errors of the variables *x, y, ...*
The partial derivatives have to be calculated at the measured values of the variables.

Example

We have a pile of resistors with a nominal resistance R_1 = (100.0 $\pm$ 2.1) Ω as calculated above, and we have another pile of resistors with a nominal resistance R_2 = (400.2 $\pm$ 4.5) Ω (this error interval is also given for P = 99% confidence level).

Assume we take one resistor from the first and one from the second pile,

and connect them in

A. series;

B. parallel.

Calculate the resultant with the error interval for the same confidence level (i.e. calculate the interval in which the resultant will be with P = 99% probability).

Calculation:

Data: $\bar{R}_1 = 100.0\ \Omega,\ \bar{R}_2 = 400.2\ \Omega,\ \Delta R_1 = 2.1\ \Omega,\ \Delta R_2 = 4.5\ \Omega.$

A. Series connection:

The function *f(x,y,...)* how the series resultant R_s is calculated from the resistances is:

$$R_s\ (R_1, R_2) = R_1 + R_2\ .$$

The mean of the series resultant is: $\bar{R}_s = \bar{R}_1 + \bar{R}_2$ = 100.0 + 400.2 = 500.2 Ω.

The partial derivative of the function $R_s(R_1,R_2) = R_1 + R_2$ with respect to R_1 and R_2, respectively:

$$\frac{\partial R_s}{\partial R_1} = \frac{\partial R_s}{\partial R_2} = 1.$$

The error interval for the series resultant:

$$\Delta R_s = \sqrt{\left(\frac{\partial R_s}{\partial R_1} \cdot \Delta R_1\right)^2 + \left(\frac{\partial R_s}{\partial R_2} \cdot \Delta R_2\right)^2} = \sqrt{(1 \cdot 2.1)^2 + (1 \cdot 4.5)^2} \approx 4.996\ \Omega.$$

So the series resultant with P = 99% probability will be in the interval

R_s = (500.2 ± 5.0) Ω.

B. Parallel connection:

The function *f(x,y,...)* how the parallel resultant R_p is calculated from the resistances is:

$$R_p\ (R_1, R_2) = \frac{R_1 \cdot R_2}{R_1 + R_2}\ .$$

The mean of the parallel resultant is:

$$\overline{R_p} = \frac{\overline{R_1} \cdot \overline{R_2}}{\overline{R_1} + \overline{R_2}} = \frac{100.0 \cdot 400.2}{100.0 + 400.2} \approx 80.008\ \Omega.$$

The partial derivative of the function $R_p(R_1, R_2) = \frac{R_1 \cdot R_2}{R_1 + R_2}$ with respect to R_1 and R_2, respectively:

$$\frac{\partial R_p}{\partial R_1} = \frac{R_2 \cdot (R_1 + R_2) - R_1 R_2 \cdot 1}{(R_1 + R_2)^2} = \left(\frac{R_2}{R_1 + R_2}\right)^2, \quad \text{and} \quad \frac{\partial R_p}{\partial R_2} = \left(\frac{R_1}{R_1 + R_2}\right)^2.$$

By substituting the values of the mean $\overline{R_1}$ = 100.0 Ω and $\overline{R_2}$ = 400.2 Ω , we get

$$\frac{\partial R_p}{\partial R_1} = \left(\frac{400.2}{100.0 + 400.2}\right)^2 \approx 0.64013 \quad \text{and} \quad \frac{\partial R_p}{\partial R_2} = \left(\frac{100.0}{100.0 + 400.2}\right)^2 \approx 0.03997.$$

The error interval for the parallel resultant:

$$\Delta R_p = \sqrt{\left(\frac{\partial R_p}{\partial R_1} \cdot \Delta R_1\right)^2 + \left(\frac{\partial R_p}{\partial R_2} \cdot \Delta R_2\right)^2} = \sqrt{(0.64013 \cdot 2.1)^2 + (0.03997 \cdot 4.5)^2} \approx 1.3562\ \Omega.$$

So the parallel resultant with P = 99% probability will be in the interval

R_p = (80.0 ± 1.4) Ω.

This result can also be written as

R_p = (80.01 ± 1.36) Ω,

because if the error starts with “1” then it is better to write 3 significant figures.

DATA SHEET

Name:

Date:

	Measured data m_i ( )	$m_i - \bar{m}$	$(m_i - \bar{m})^2$
1.			
2.			
3.			
4.			
5.			
6.			
7.			

n =

Arithmetic mean: $\bar{m} =$

Standard deviation of the mean: $s_{\bar{m}} =$

Confidence level: P = 90%

Student's parameter: t =

Confidence interval: $\Delta m =$

Result: $m = (\qquad \pm \qquad)$