1. Determining the spring constant *k* based on elongation

THEORY

According to Hooke's Law the restoring force of an ideal spring is proportional to its <u>elongation</u> x:

$F = k \cdot x$

where $x = \ell - \ell_0$, ℓ is the actual length of the spring, ℓ_0 is its length in relaxed state;

k is the spring constant (or stiffness; dimension: $\frac{\mathrm{N}}{\mathrm{m}} = \frac{\mathrm{kg}}{\mathrm{s}^2}$).

For an ideal spring there is no force when $\ell = \ell_0$, i.e. $x = 0$.

When a mass m is suspended at the end of the spring in vertical position, in equilibrium

$k \cdot x = m \cdot g.$

The spring constant can be determined by measuring the length of the spring with different loads.

MEASUREMENT

Tools: a spring in vertical position, a mm scale along it, nuts as known masses, a PVC rod as a holder of the nuts, and an object with unknown mass.

First measure m_{PVC}, the mass of the PVC rod (you may also check m_{nut}, the mass of the nuts), then read the lowest position of the spring (denoted by z in the table)

- without any load;
- with the PVC rod;
- with 1 nut; with 2 nuts; etc;
- with the object with unknown mass.

EVALUATION – theory

The mass of the load can be expressed in terms of the number of the nuts: $m = m_{PVC} + n \cdot m_{nut}$.

The elongation x can be expressed in terms of the measured position z: $x = z - z_0$, where z_0 is the position without any load,

so $k \cdot x = m \cdot g \quad \rightarrow \quad k \cdot (z - z_0) = (m_{PVC} + n \cdot m_{nut}) \cdot g$

We expect that by plotting z *vs.* n the depicted points fit to a straight line, but we will see that in this case it not true. In our case $x = 0 \rightarrow F = 0$ is not valid. The reason for this is that there is a force F_0 acting even in the relaxed spring. This has to be included in the equation:

$k \cdot x + F_0 = m \cdot g$

$k \cdot (z - z_0) + F_0 = (m_{PVC} + n \cdot m_{nut}) \cdot g$ [Eq 1]

Neglecting the points that do not fit to the straight line because of force F_0 the spring constant k can be calculated from the slope using [Eq1].

The slope of the straight line can be calculated by graphical method and by linear regression.

<u>Graphical method:</u>

The slope of the line is $a = \frac{\Delta z}{\Delta n} = \frac{z_2 - z_1}{n_2 - n_1}$ calculated from the points (n_1, z_1) and (n_2, z_2).

Substituting these two points into [Eq 1]

$k \cdot (z_1 - z_0) + F_0 = (m_{PVC} + n_1 \cdot m_{nut}) \cdot g$

$k \cdot (z_2 - z_0) + F_0 = (m_{PVC} + n_2 \cdot m_{nut}) \cdot g$

the difference gives

$k \cdot (z_2 - z_1) = (n_2 - n_1) \cdot m_{nut} \cdot g$.

The spring constant can be calculated as

$$k = \frac{n_2 - n_1}{z_2 - z_1} \cdot m_{nut} \cdot g = \frac{\Delta n}{\Delta z} \cdot m_{nut} \cdot g = \left(\frac{\Delta z}{\Delta n}\right)^{-1} \cdot m_{nut} \cdot g = \frac{1}{a} \cdot m_{nut} \cdot g \ .$$

Method of least squares:
The measured position *z* can be expressed from [Eq 1] as a function of the number of the nuts *n*:

$$F_0 + k\cdot(z-z_0) = (m_{PVC} + n\cdot m_{nut})\cdot g \quad \rightarrow \quad z = \frac{m_{nut}\cdot g}{k}\cdot n + \left(\frac{m_{PVC}\cdot g - F_0}{k} + z_0\right)$$

z is a linear function of *n* with a slope of $a = \frac{m_{nut}\cdot g}{k}$.

EVALUATION – HOMEWORK:

1. Plot the position of the lowest point of the spring *z* against the number of the nuts *n* on an A4 size mm paper.
2. Based on the graph find the values that fit to a straight line.
3. Determine the unknown mass using the diagram.
4. Read the coordinates of two distant points of the straight line and calculate the slope $\frac{\Delta z}{\Delta n}$.
5. Calculate the spring constant *k*.
6. Evaluate the measurement using the method of least squares:
   choose the values that fit to the line;
   calculate the averages $\bar{n}, \bar{z}, \overline{n^2}, \overline{n\cdot z}$;
   calculate the slope;
   calculate the value of the spring constant.

2. Oscillations of a spring

THEORY

Undamped oscillations

When a mass *m* is suspended at the end of the spring in <u>horizontal</u> position (neglecting friction) then the equation of motion is

$$m\cdot d^2x/dt^2 = -k\cdot x\ .$$

$k\cdot x$ is negative because when $x > 0$ (i.e. when $\ell > \ell_0$) the force is negative, and when $x < 0$ (i.e. when $\ell < \ell_0$), the force is positive, as the force points always towards the relaxed state $x = 0$ (so the direction of the restoring force is always opposite to that of the elongation).

By displacing the mass from the relaxed state $x = 0$ to an initial position $x = x_0$ and/or giving an initial speed v_0 to the mass, it will exhibit simple harmonic motion:

$$x(t) = A\cdot\cos(\omega\cdot t + \varphi_0)$$

where *A* is the amplitude (the maximal elongation)
  ω is the angular frequency [s^{-1}] : $\omega = 2\pi f$, where
    f is the frequency: $f = 1/T$ [Hz], where
      T is the time period [s];
  $\omega\cdot t + \varphi_0$ is the phase of the oscillations;
  φ_0 is the phase constant [rad].

The speed of the mass is the derivative of $x(t)$:

$v(t) = dx/dt = -A\cdot\omega\cdot\sin(\omega\cdot t + \varphi_0)$;   the maximal speed (when x = 0): $v_{max} = A\cdot\omega$.

The acceleration is the derivative of the speed $v(t)$:

$a(t) = dv/dt = d^2x/dt^2 = -A\cdot\omega^2\cdot\cos(\omega\cdot t + \varphi_0)$; observe that $a(t) = -\omega^2\cdot x(t)$.

Comparing $d^2x/dt^2 = -(k/m)\cdot x$ from the equation of motion
and $d^2x/dt^2 = -\omega^2\cdot x$ from the acceleration, we can see that $\omega^2 = k/m$, so

the angular frequency ω is determined by the mass *m* and the spring constant *k*:

$\omega = \sqrt{k/m}$, and hence the time period *T*

$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}.$$

The amplitude *A* and the initial phase φ_0 are determined by the initial conditions x_0 and v_0:

$$\varphi_0 = \text{arc tg}\left(-\frac{v_0}{\omega\cdot x_0}\right), \quad A = \sqrt{\left(\frac{v_0}{\omega}\right)^2 + {x_0}^2} \quad .$$

Damped oscillations

If there is also a damping force acting on the body that is proportional to the speed, we have the following equation of motion:

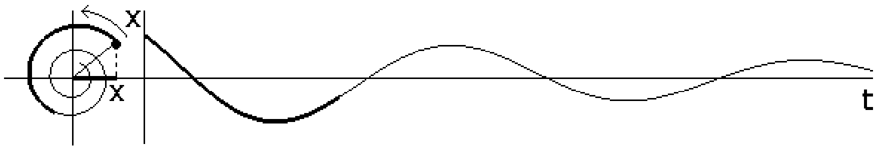
$$m\cdot d^2x/dt^2 = -k\cdot x - c\cdot dx/dt.$$

For relatively small damping factor *c* the solution is similar to the simple harmonic motion but with exponentially decreasing amplitude:

$$x(t) = A_0\cdot e^{-\beta\cdot t}\cdot\cos(\omega\cdot t + \varphi_0).$$

The angular frequency ω is smaller than that of the undamped oscillation (ω_0):

$\omega = \sqrt{{\omega_0}^2 - \beta^2}$, where $(\omega_0)^2 = k/m$ and $2\beta = c/m$.



In the case of higher damping factor (i.e. when $\beta \geq \omega_0$) the motion gets aperiodic.

The spring in vertical position (undamped oscillations)

Here we must also consider the gravitational force:

$$m\cdot d^2y/dt^2 = -k\cdot(y-\ell_0) + mg$$

ℓ_0 is the length of the spring and *y* is the position of the body measured from the end of the spring. At first sight this equation of motion is different from the equation for the horizontal case. However, by introducing a new variable it can be seen that its solution is also a simple harmonic oscillation but the equilibrium point is shifted.

The stable equilibrium position y_E of the body is calculated from the condition that $d^2y/dt^2 = 0$ and it is

$$y_E = \ell_0 + mg/k.$$

The time period of the spring in vertical position equals to that of in horizontal position, only the equilibrium position changes from ℓ_0 to $\ell_0 + mg/k$.

MEASUREMENT

Put three different loads (e.g. 4, 7, 10 nuts) on the PVC rod, suspend it to the spring, pull it, let it oscillate and measure the time of 10 periods.

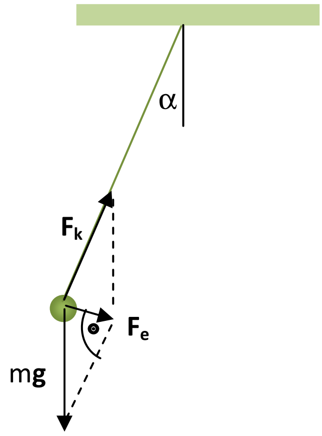
EVALUATION – HOMEWORK:

Calculate the spring constant for all cases, using the formula for the time period *T* for undamped oscillations.

Compare the 1+3 values obtained for the spring constant.

3. The simple pendulum

THEORY



The equation of motion of a small object *m* suspended from the end of a light, inextensible string of length *L* swinging in a <u>vertical</u> plane is the following:

$$m \cdot a_t = m \cdot L \frac{d^2\alpha}{dt^2} = -m \cdot g \cdot \sin\alpha$$

where α is the time dependent angle of the string measured from the equilibrium (vertical) position. As it is a nonlinear differential equation we introduce the following approximation:

$$\sin\alpha \approx \alpha$$

and so the problem is similar to that of the spring:

$$m \cdot L \frac{d^2\alpha}{dt^2} = -m \cdot g \cdot \alpha \quad \rightarrow \quad \frac{d^2\alpha}{dt^2} = -(g/L) \cdot \alpha$$

$$\alpha(t) = \alpha_{max} \cos(\omega t + \varphi_0) \quad \rightarrow \quad \frac{d^2\alpha}{dt^2} = -\omega^2 \cdot \alpha$$

$$\rightarrow \qquad \omega = \sqrt{\frac{g}{L}} = \frac{2\pi}{T}$$

This means that the motion of the pendulum is harmonic with a time period of

$$\boxed{T = 2\pi\sqrt{\frac{L}{g}}}\ .$$

The deviation caused by the approximation $\sin\alpha \approx \alpha$ is
0.05% for α=5°, 1% for α=22° and 18% for α=90°.

MEASUREMENT

Measure the length of the string *L*.
Set the pendulum bob in motion with a <u>small</u> initial angle.
Measure the time of 10 periods.
Repeat the measurement 5 times.
Measure the time of 10 periods when the initial angle is increased.

EVALUATION – HOMEWORK

1. Calculate the average time period $\overline{T}$ and the confidence interval ΔT for P = 95%.
2. Calculate the mean value of the acceleration of gravity $\bar{g}$ (using the value $\overline{T}$).
3. Calculate the confidence interval Δg using the formula of error propagation.
   ΔL is not measured, use 3–5 mm as an estimation.