1. Determining the spring constant *k* based on elongation

THEORY

According to Hooke's Law the restoring force of an ideal spring is proportional to its <u>elongation</u> x:

$F = k \cdot x$

where $x = \ell - \ell_0$, ℓ is the actual length of the spring, ℓ_0 is its length in relaxed state;

k is the spring constant (or stiffness; dimension: $\frac{\mathrm{N}}{\mathrm{m}} = \frac{\mathrm{kg}}{\mathrm{s}^2}$).

For an ideal spring there is no force when $\ell = \ell_0$.

When a mass m is suspended at the end of the spring in vertical position, in equilibrium

$k \cdot x = m \cdot g.$

The spring constant can be determined by measuring the length of the spring with different loads.

MEASUREMENT

We have a spring in vertical position, a mm scale along it, nuts (female screws) as known masses, a PVC rod as a holder of the nuts, and an unknown mass.

Measure the mass of the PVC rod. (You may also check the mass of the nuts.)

Measure the lowest position of the spring (z)

- without any load;
- with the PVC rod;
- with 1 nut; with 2 nuts; with … nuts;
- with the object with unknown mass.

EVALUATION

1. Plot the position of the lowest point of the spring z against the number of the nuts n.

From the diagram it can be seen that in this case $F = k \cdot x$ is not valid. The reason for this is that there is a force F_0 acting even in the relaxed spring. This has to be included in the equation:

$k \cdot x + F_0 = m \cdot g$

The mass of the load can be expressed in terms of the number of the nuts: $m = m_{PVC} + n \cdot m_{nut}$.

We can avoid calculating the elongation and simply use the measured position z instead:

$x = z - z_0$, where z_0 is the position with the PVC rod, so

$k \cdot (z - z_0) + F_0 = (m_{PVC} + n \cdot m_{nut}) \cdot g$ [Eq 1]

2. The spring constant k can be determined from the slope of the diagram with the following two methods:

<u>Graphical method:</u>

Taking two loads, n_1 and n_2:

$k \cdot (z_1 - z_0) + F_0 = (m_{PVC} + n_1 \cdot m_{nut}) \cdot g$

$k \cdot (z_2 - z_0) + F_0 = (m_{PVC} + n_2 \cdot m_{nut}) \cdot g$

The difference gives

$k \cdot (z_2 - z_1) = (n_2 - n_1) \cdot m_{nut} \cdot g$

so the spring constant can be calculated as

$$k = \frac{n_2 - n_1}{z_2 - z_1} \cdot m_{nut} \cdot g = \frac{\Delta n}{\Delta z} \cdot m_{nut} \cdot g$$

Read the slope $\frac{\Delta z}{\Delta n}$ from the graph and calculate k.

Method of least squares:

The measured position z can be expressed from [Eq 1] as a function of the number of the nuts n:

$$F_0 + k\cdot(z - z_0) = (m_{PVC} + n\cdot m_{nut})\cdot g \quad \rightarrow \quad z = \frac{m_{nut}\cdot g}{k}\cdot n + \left(\frac{m_{PVC}\cdot g - F_0}{k} + z_0\right)$$

which is a linear function with a slope = $\frac{m_{nut}\cdot g}{k}$.

> For the linear equation $y = a\cdot x + b$ using the least squares method
>
> the slope is calculated as $a = \frac{\overline{x\cdot y} - \bar{x}\cdot\bar{y}}{\overline{x^2} - \bar{x}^2}$.

In our case now $x = n$, $y = z$, $a = \frac{m_{nut}\cdot g}{k}$, and $b = \frac{m_{PVC}\cdot g - F_0}{k} + z_0$.

so $a = \frac{\overline{n\cdot z} - \bar{n}\cdot\bar{z}}{\overline{n^2} - \bar{n}^2} = \frac{m_{nut}\cdot g}{k}$.

Based on the graph find the values that fit on a straight line. Using these values calculate the slope with the least squares method. Calculate the value of the spring constant.

3. Determine the unknown mass using the diagram.

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2. Oscillations of a spring

THEORY

Undamped oscillations

When a mass m is suspended at the end of the spring in horizontal position (neglecting friction) then the equation of motion is

$$m\cdot d^2x/dt^2 = -\,k\cdot x\,.$$

$k\cdot x$ is negative because when $x > 0$ (i.e. when $\ell > \ell_0$) the force is negative, and when $x < 0$ (i.e. when $\ell < \ell_0$), the force is positive, as the force points always towards the relaxed state $x = 0$ (so the direction of the restoring force is always opposite to that of the elongation).

By displacing the mass from the relaxed state $x = 0$ to an initial position $x = x_0$ and/or giving an initial speed v_0 to the mass, it will exhibit simple harmonic motion:

$$x(t) = A\cdot\cos(\omega\cdot t + \varphi_0)$$

where A is the amplitude (the maximal elongation)

ω is the angular frequency [s^{-1}] : $\omega = 2\pi f$, where

f is the frequency: $f = 1/T$ [Hz], where

T is the time period [s];

$\omega\cdot t + \varphi_0$ is the phase of the oscillations;

φ_0 is the phase constant [rad].

The speed of the mass is the derivative of $x(t)$:

$v(t) = dx/dt = -A\cdot\omega\cdot\sin(\omega\cdot t + \varphi_0)$; the maximal speed (when x = 0): $v_{max} = A\cdot\omega$.

The acceleration is the derivative of the speed $v(t)$:

$a(t) = dv/dt = d^2x/dt^2 = -A\cdot\omega^2\cdot\cos(\omega\cdot t + \varphi_0)$; observe that $a(t) = -\omega^2\cdot x(t)$.

Comparing $d^2x/dt^2 = -\,(k/m)\cdot x$ from the equation of motion

and $d^2x/dt^2 = -\omega^2\cdot x$ from the acceleration, we can see that $\omega^2 = k/m$, so the angular frequency ω is determined by the mass m and the spring constant k:

$\omega = \sqrt{k/m}$, and hence the time period T

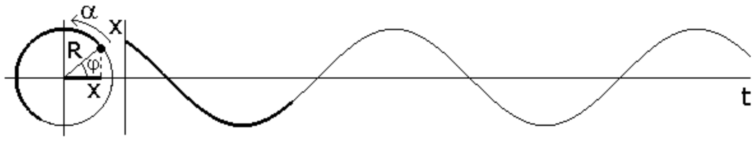
$$\boxed{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}}\,.$$

The amplitude A and the initial phase φ_0 are determined by the initial conditions x_0 and v_0:

$$\varphi_0 = arc\,tg\left(-\frac{v_0}{\omega \cdot x_0}\right), \qquad A = \sqrt{\left(\frac{v_0}{\omega}\right)^2 + x_0^{\ 2}}\ .$$

Harmonic oscillation is obtained also by the projection of a <u>uniform</u> circular motion onto the *x* axis:

$x(t) = A\cdot cos(\omega t + \varphi_0)$ (here the radius *R* is replaced by the amplitude *A*)



Damped oscillations

If there is also a damping force acting on the body that is <u>proportional to the speed</u>, we have the following equation of motion:

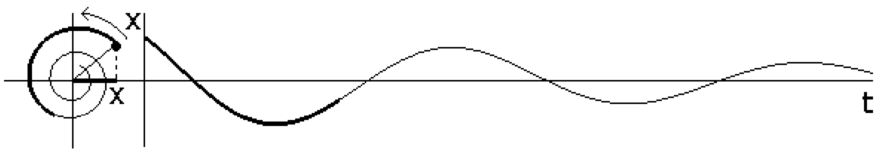
$m\cdot d^2x/dt^2 = -\,k\cdot x - c\cdot dx/dt.$

For relatively small damping factor *c* the solution is similar to the simple harmonic motion but with exponentially decreasing amplitude:

$x(t) = A_0\cdot e^{-\beta\cdot t}\cdot cos(\omega\cdot t + \varphi_0).$

The angular frequency ω is smaller than that of the undamped oscillation (ω_0):

$\omega = \sqrt{\omega_0^{\ 2} - \beta^2}$, where $(\omega_0)^2 = k/m$ and $2\beta = c/m$.



In the case of higher damping factor (i.e. when $\beta \geq \omega_0$) the motion gets aperiodic.

The spring in vertical position (undamped oscillations)

Here we must also consider the gravitational force:

$m\cdot d^2y/dt^2 = -\,k\cdot(y-\ell_0) + mg$

ℓ_0 is the length of the spring and *y* is the position of the body measured from the end of the spring. At first sight this equation of motion is different from the equation for the horizontal case. However, by introducing a new variable it can be seen that its solution is also a simple harmonic oscillation but the equilibrium point is shifted.

The stable equilibrium position y_E of the body is calculated from the condition that $d^2y/dt^2 = 0$ and it is

$y_E = \ell_0 + mg/k.$

The time period of the spring in vertical position equals to that of in horizontal position, only the equilibrium position changes from ℓ_0 to $\ell_0 + mg/k.$

MEASUREMENT

Put three different loads (e.g. 4, 8, 12 nuts) on the PVC rod, suspend it to the spring, pull it, let it oscillate and measure the time of 10 periods.

EVALUATION

Calculate the spring constant for all cases, using the formula for the time period *T* for undamped oscillations.

Compare the 1+3 values obtained for the spring constant.